CHAPTER

7 CALCULUS OF THE NATURAL LOGARITHMIC FUNCTION

SYLLABUS COVERAGE

Syllabus coverage

TOPIC 3.1 FURTHER DIFFERENTIATION AND APPLICATIONS

The second derivative and applications of differentiation

3.1.10 use the increments formula: $\delta y = \frac{dy}{dx} \times \delta x$ to estimate the change in the dependent variable y resulting from changes in the independent variable x

3.1.11 apply the concept of the second derivative as the rate of change of the first derivative function

3.1.12 identify acceleration as the second derivative of position with respect to time

3.1.13 examine the concepts of concavity and points of inflection and their relationship with the second derivative

3.1.14 apply the second derivative test for determining local maxima and minima

3.1.15 sketch the graph of a function using first and second derivatives to locate stationary points and points of inflection

3.1.16 solve optimisation problems from a wide variety of fields using first and second derivatives

TOPIC 3.2 INTEGRALS

Applications of integration

3.2.19 calculate the area under a curve

3.2.20 calculate the area between curves determined by functions of the form $y = f(x)$

3.2.21 determine displacement given velocity in linear motion problems

3.2.22 determine positions given linear acceleration and initial values of position and velocity

TOPIC 4.1 THE LOGARITHMIC FUNCTION

Calculus of the natural logarithmic function

4.1.9 define the natural logarithm $\ln x = \log_e x$

4.1.10 examine and use the inverse relationship of the functions $y = e^x$ and $y = \ln x$

4.1.11 establish and use the formula $\frac{d}{dx}\ln x = \frac{1}{x}$

4.1.12 establish and use the formula $\int \frac{1}{x}\,dx = \ln x + c$, for $x > 0$

4.1.13 determine derivatives of the form $\frac{d}{dx}(\ln f(x))$ and integrals of the form $\int \frac{f'(x)}{f(x)}\,dx$, for $f(x) > 0$

4.1.14 use logarithmic functions and their derivatives to solve practical problems

Video playlists (5):

- **7.1** Differentiating natural logarithmic functions
- **7.2** Applications of derivatives of the natural logarithmic function
- **7.3** Integrals producing natural logarithmic functions
- **7.4** Applications of anti-differentiation involving the natural logarithms

WACE question analysis Calculus of the natural logarithmic function

Worksheets (4):

- **7.1** Derivatives of logarithmic functions
  - Exponential and logarithmic functions
  - Differentiating exponential and logarithmic functions
- **7.3** Integration of $\frac{1}{x}$

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To access resources above, visit **cengage.com.au/nelsonmindtap**

Video playlist
Differentiating natural logarithmic functions

Worksheets
Derivatives of logarithmic functions

Exponential and logarithmic functions

7.1 Differentiating natural logarithmic functions

The first derivative of ln(x)

The first derivative of $\ln(x)$ can be found using the algebraic property $e^{\ln x} = x$.

Differentiate both sides. $\frac{d}{dx}\left(e^{\ln x}\right) = \frac{d}{dx}(x)$

Using the chain rule: $\frac{d}{dx}(\ln x)e^{\ln x} = 1$

$$\frac{d}{dx}(\ln x) \times x = 1$$

Therefore, $\frac{d}{dx}(\ln x) = \frac{1}{x}$

The first derivative of ln(x)

$$\frac{d}{dx}(\ln(x)) = \frac{1}{x}$$

The first derivative of ln(ax)

Using the logarithm law: $\ln(ax) = \ln(a) + \ln(x)$

$$\frac{d}{dx}(\ln(ax)) = \frac{d}{dx}(\ln(a)) + \frac{d}{dx}(\ln(x))$$

and as $\ln(a)$ is a constant, $\frac{d}{dx}(\ln(a)) = 0$

$$\frac{d}{dx}(\ln(ax)) = \frac{1}{x}$$

The first derivative of ln(ax)

$$\frac{d}{dx}(\ln(ax)) = \frac{1}{x}$$

The first derivative of ln($f(x)$)

We can use the chain rule to create a formula for the derivative of the natural logarithm of a function $f(x)$.

If $y = \ln(f(x))$,

then $\frac{dy}{dx} = \frac{1}{f(x)} \times f'(x) = \frac{f'(x)}{f(x)}$.

The chain rule for natural logarithmic functions

If $y = \ln(f(x))$, then $\frac{d}{dx}(\ln(f(x)) = \frac{f'(x)}{f(x)}$.

7.1

WORKED EXAMPLE 1 **Finding the derivative of $y = \ln(f(x))$**

Find the first derivative of each logarithmic function.

a $y = \ln(3x - 7)$

b $y = \ln(9x^2 - x)$

Steps	Working
a **1** Use the rule $\frac{d}{dx}(\ln(f(x)) = \frac{f'(x)}{f(x)}$.	$f(x) = 3x - 7$ $f'(x) = 3$
2 Let $f(x) = \ln(3x - 7)$ and differentiate.	$\frac{d}{dx}(\ln(3x - 7)) = \frac{3}{3x - 7}$
b Let $f(x) = 9x^2 - x$ and differentiate.	$f(x) = 9x^2 - x$ $f'(x) = 18x - 1$ $\frac{d}{dx}(\ln(9x^2 - x)) = \frac{18x - 1}{9x^2 - x}$

Some derivatives of natural logarithmic functions are much easier if the logarithm is first simplified using the laws of logarithms.

The laws of logarithms for natural logarithms

$\ln(xy) = \ln(x) + \ln(y)$

$\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$

$\ln(x^n) = n\ln(x)$

Also remember,

$\ln(1) = 0$

$\ln(e^n) = n$

WORKED EXAMPLE 2 **Using the laws of logarithms to find the first derivative of natural logarithmic functions**

Find the first derivative of each logarithmic function.

a $y = \ln(5x)$

b $y = \ln((2x - 5))^2$

c $y = \ln(\sqrt{x})$

d $y = \ln((x + 2)(x + 5))$

Steps	Working
a Use the rule $\frac{d}{dx}(\ln(ax)) = \frac{1}{x}$.	$\frac{d}{dx}(\ln(5x)) = \frac{1}{x}$
b **1** Simplify using the logarithm law $\ln(x^n) = n\ln(x)$.	$y = \ln((2x - 5)^2)$ $y = 2\ln(2x - 5)$
2 Use the chain rule $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.	$u = 2x - 5$ $\quad y = 2\ln(u)$ $\frac{du}{dx} = 2$ $\quad \frac{dy}{du} = \frac{2}{u} = \frac{2}{2x - 5}$ $\frac{dy}{dx} = 2 \times \frac{2}{2x - 5} = \frac{4}{2x - 5}$

c Simplify using the logarithm laws and differentiate.	$y = \ln(\sqrt{x}) = \ln\left(x^{\frac{1}{2}}\right)$ $y = \frac{1}{2}\ln(x)$ $\frac{dy}{dx} = \frac{1}{2} \times \frac{1}{x}$ $\frac{dy}{dx} = \frac{1}{2x}$
d Simplify using logarithm laws and differentiate.	$y = \ln((x+2)(x+5))$ $y = \ln(x+2) + \ln(x+5)$ $\frac{dy}{dx} = \frac{1}{x+2} + \frac{1}{x+5}$

WORKED EXAMPLE 3 Finding the first derivative using the product rule

Find the first derivative of $y = x^2 \ln(x)$.

Steps	Working
1 Identify u and v.	$u = x^2$ $\quad$ $v = \ln(x)$
2 Differentiate to obtain $\frac{du}{dx}$ and $\frac{dv}{dx}$.	$\frac{du}{dx} = 2x$ $\quad$ $\frac{dv}{dx} = \frac{1}{x}$
3 Use the product rule $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$ and simplify.	$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$ $= x^2 \times \frac{1}{x} + \ln(x) \times 2x$ $= x + 2x\ln(x)$

WORKED EXAMPLE 4 Finding the first derivative using the quotient rule

Find the first derivative of $f(x) = \frac{\ln(x)}{x^2}$.

Steps	Working
1 Let $\frac{u}{v} = \frac{\ln(x)}{x^2}$.	$u = \ln(x)$ $\quad$ $v = x^2$ $\frac{du}{dx} = \frac{1}{x}$ $\quad$ $\frac{dv}{dx} = 2x$
2 Differentiate using the quotient rule $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$.	$f'(x) = \frac{x^2 \times \frac{1}{x} - 2x\ln(x)}{(x^2)^2}$ $= \frac{x - 2x\ln(x)}{x^4}$ $= \frac{1 - 2\ln(x)}{x^3}$

Finding the second derivative of a natural logarithmic function

The second derivative of a function is the rate of change of the first derivative. This can be used to find the rate at which the gradient of a function is changing. It is also used to find points of inflection and to determine the nature of a stationary point.

WORKED EXAMPLE 5 Finding the second derivative of a natural logarithmic function

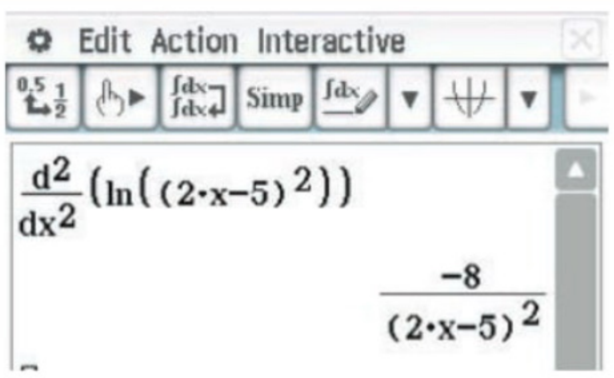
Find the second derivative of $y = \ln(2x - 3)$.

Steps	Working
1 Find the first derivative.	Let $f(x) = 2x - 3$. $f'(x) = 2$ $\frac{d}{dx}(\ln(2x - 3)) = \frac{2}{2x - 3}$
2 Write $\frac{2}{2x - 3}$ as $2(2x - 3)^{-1}$ and differentiate to find the second derivative.	$\frac{2}{2x - 3} = 2(2x - 3)^{-1}$ $\frac{d^2y}{dx^2} = -2(2x - 3)^{-2} \times 2$ $= -4(2x - 3)^{-2}$ $= \frac{-4}{(2x - 3)^2}$

USING CAS 1 Finding the second derivative of a natural logarithmic function

Find the second derivative of $y = \ln(2x - 5)^2$.

ClassPad



1 Enter and highlight the expression.
2 Tap **Interactive** > **Calculus** > **diff**.
3 Enter **2** as the order.

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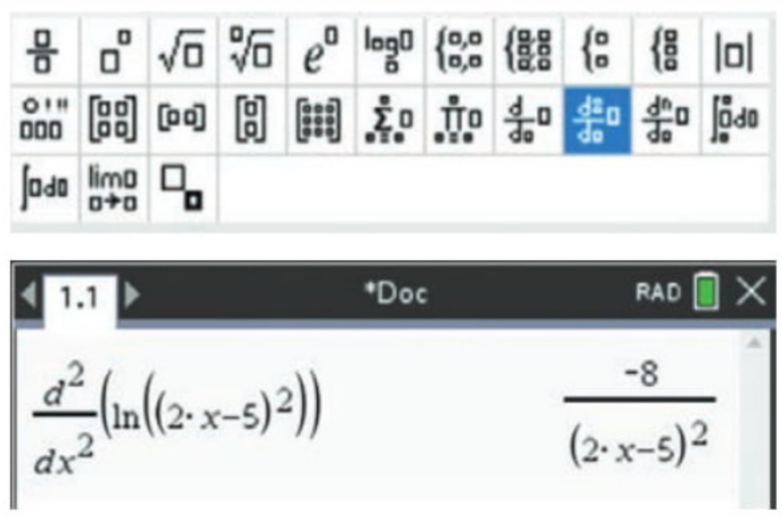


1 Press the **maths template** and select the second derivative.
2 Enter the expression, including the **dx**.

The second derivative is $-\frac{8}{(2x - 5)^2}$.

EXERCISE 7.1 Differentiating natural logarithmic functions

ANSWERS p. 401

Mastery

1 WORKED EXAMPLE 1 Find $\frac{dy}{dx}$ for each of the natural logarithm functions below.

a $y = \ln(8x - 5)$ **b** $y = \ln(3x^2 + 6x)$ **c** $y = 3\ln(x^4 + 8x)$

2 WORKED EXAMPLE 2 Find the derivative of each function using the laws of logarithms to simplify where necessary.

a $y = \ln(2x)$ **b** $y = 3\ln(5x)$ **c** $y = 2\ln(4x - 3)$

d $y = \ln(\sqrt[4]{x - 4})$ **e** $y = \ln((2x + 1)^3)$

3 WORKED EXAMPLE 3 Find $f'(x)$ for each function.

a $f(x) = (x^2 - 2x)\ln(x)$ **b** $f(x) = x^3\ln(x^3)$ **c** $f(x) = \frac{1}{x}\ln(x)$

4 WORKED EXAMPLE 4 Find $f'(x)$ if $f(x) = \frac{\ln(2x)}{x^3}$.

5 WORKED EXAMPLE 5 Find the second derivatives of the following natural logarithmic functions.

a $y = \ln(5x + 4)$ **b** $y = 2\ln((4x + 1)^2)$

6 Using CAS 1 Given $f(x) = \ln(4x - 3)$, find

a $f'(x)$ **b** $f''(x)$.

Calculator-free

7 (2 marks) Find the first derivative of $f(x) = \sin(\ln(x^2))$ at $x = e$.

8 (4 marks)

a Show that $\ln\sqrt{\frac{3x + 3}{3x - 2}} = \frac{1}{2}\ln(3x + 3) - \frac{1}{2}\ln(3x - 2)$. (3 marks)

b Hence find the first derivative of $f(x) = \ln\left(\sqrt{\frac{3x + 3}{3x - 2}}\right)$ at $x = 2$. (1 mark)

9 (2 marks) If $y = x^2\ln(x)$, find $\frac{dy}{dx}$.

10 (2 marks) Differentiate $x\ln(x)$ with respect to x.

11 (3 marks) Let $f(x) = \frac{\ln(x)}{x^2}$.

a Find $f'(x)$. (2 marks)

b Evaluate $f'(1)$. (1 mark)

12 (2 marks) For $f(x) = \log_e(x^2 + 1)$, find $f'(2)$.

Calculator-assumed

13 © SCSA MM2016 Q13a MODIFIED (2 marks) Determine $\frac{d}{dx}(x^3\ln(2x))$.

7.2 Applications of derivatives of the natural logarithmic function

Stationary points and their nature

Local maxima occur when $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$.

Local minima occur when $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$.

Stationary points of inflection occur when $\frac{dy}{dx} = 0$, $\frac{d^2y}{dx^2} = 0$ and the curve either changes from concave up to concave down or concave down to concave up. A point on a curve where the concavity changes is called a point of inflection and satisfies the same conditions as a stationary point of inflection; however, it is not a stationary point so $\frac{dy}{dx}$ is NOT equal to zero.

Video playlist Applications of derivatives of the natural logarithmic function

WORKED EXAMPLE 6 Finding the coordinates and nature of a local maximum

The function $f(x) = \ln(10x - x^2)$ has a stationary point in the interval $0 < x < 10$.

a Find the coordinates of the stationary point.

b Use the second derivative to determine the nature of the stationary point.

Steps	Working
a 1 Find $f'(x)$. Use the rule $\frac{d}{dx}(\ln(f(x)) = \frac{f'(x)}{f(x)}$.	$f'(x) = \frac{10 - 2x}{10x - x^2}$
2 Solve $f'(x) = 0$.	Stationary when $f'(x) = 0$. $\frac{10 - 2x}{10x - x^2} = 0$ $10 - 2x = 0$ $x = 5$
3 Find $f(5)$ and state the coordinates of the stationary point.	$f(5) = \ln(10 \times 5 - 5^2) = \ln(25)$ Stationary point is $(5, \ln(25))$.
b 1 Find the second derivative by differentiating $f'(x)$ using the quotient rule.	$u = 10 - 2x$ $v = 10x - x^2$ $\frac{du}{dx} = -2$ $\frac{dv}{dx} = 10 - 2x$ $f''(x) = \frac{-2(10x - x^2) - (10 - 2x)(10 - 2x)}{(10x - x^2)^2}$ $= \frac{-20x + 2x^2 - 100 + 40x - 4x^2}{(10x - x^2)^2}$ $= \frac{-2x^2 + 20x - 100}{(10x - x^2)^2}$

2 Find the nature of the stationary point by finding $f''(5)$.	$f''(5) = \dfrac{-2(5)^2 + 20(5) - 100}{(10(5) - (5)^2)^2}$ $= \dfrac{-50}{625} = -\dfrac{2}{25}$ The stationary point $(5, \ln 25)$ is a local maximum as $f''(5) < 0$.

The first derivative can also be used to find an optimum solution for a function which may be the minimum production cost or maximum population number.

WORKED EXAMPLE 7 Finding the optimum solution for a natural logarithmic function

The population of tadpoles in a dam is recorded each week for eight weeks. The number of tadpoles N, after t weeks, is modelled by the function $N(t) = 100\ln(-t^2 + 8t + 9)$.

Find

a $N'(t)$

b the number of weeks when the population of tadpoles is a maximum

c the maximum population of tadpoles.

Steps	Working
a Find $N'(t)$. Use the rule $\dfrac{d}{dx}(\ln(f(x)) = \dfrac{f'(x)}{f(x)}$.	$N'(t) = \dfrac{100(8 - 2t)}{-t^2 + 8t + 9}$
b Solve $N'(t) = 0$.	Stationary when $N'(t) = 0$. $100(8 - 2t) = 0$ $t = 4$ The population of tadpoles is a maximum at 4 weeks.
c Find $N(4)$ and round to the nearest integer.	$N(4) = 100\ln(-(4)^2 + 8(4) + 9)$ $N(4) = 100\ln(25) \approx 322$ tadpoles

WORKED EXAMPLE 8 Finding the equation of the tangent

Find the equation of the tangent to the curve $f(x) = \ln(2x + e)$ at $x = 0$.

Steps	Working
1 Differentiate $f(x)$.	$f'(x) = \dfrac{2}{2x + e}$
2 Find $f'(0)$ and $f(0)$.	$f'(0) = \dfrac{2}{2(0) + e} = \dfrac{2}{e}$ $f(0) = \log_e(2(0) + e) = \log_e(e) = 1$
3 Use the formula $y - y_1 = m(x - x_1)$ to find the equation of the tangent.	$m = \dfrac{2}{e}$ The point $(0, 1)$ is on the curve $f(x)$. $y - 1 = \left(\dfrac{2}{e}\right)(x - 0)$ $y = \left(\dfrac{2}{e}\right)x + 1$

The increments formula

The increments formula can be used to approximate the increase in the y value for a corresponding small increase in the x value.

$$\delta y \approx \frac{dy}{dx} \times \delta x$$

For a given function $y = f(x)$, we can use δy to find an approximation for $f(x + \delta x)$.

$$f(x + \delta x) \approx f(x) + \delta y$$

WORKED EXAMPLE 9 Applying the increments formula

Given that $\ln(5) \approx 1.609$, use the increments formula to determine an approximation for $\ln(5.01)$.

Steps	Working
1 Find the first derivative of $\ln(x)$.	$y = \ln(x)$ $\frac{dy}{dx} = \frac{1}{x}$
2 Find the values of x and δx.	x increases from 5 to 5.01, therefore, $x = 5$ and $\delta x = 0.01$.
Find the value of $\frac{dy}{dx}$ at the given x value.	When $x = 5$ $\frac{dy}{dx} = \frac{1}{5}$.
3 Substitute into $\delta y \approx \frac{dy}{dx} \times \delta x$.	$\delta y \approx \frac{1}{5} \times 0.01 = 0.002$
4 Substitute into $f(x + \delta x) \approx f(x) + \delta y$.	$f(x) = \ln(x)$ $f(x + \delta x) \approx f(x) + \delta y$ $f(5.01) \approx f(5) + 0.002$ $\approx 1.609 + 0.002$ Therefore, $\ln(5.01) \approx 1.611$.

Straight line motion and the natural logarithmic function

Displacement: $x(t)$

Velocity: $v(t) = \frac{dx}{dt}$

Acceleration: $a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2}$

WORKED EXAMPLE 10 Straight line motion

The distance covered by Ali on her morning run is given by the function $x(t) = 5\ln(4t + 1)$, where x is the number of kilometres travelled in t hours. Find

a Ali's running speed at time t hours

b the number of hours Ali has been running when her speed is 10 km/h

c Ali's acceleration after she has been running for 2 hours.

Steps	Working
a Find the first derivative. $v(t) = x'(t)$	$x'(t) = \frac{5 \times 4}{4t + 1}$ $v(t) = \frac{20}{4t + 1}$ km/h
b Solve $x'(t) = 10$.	$\frac{20}{4t + 1} = 10$ $20 = 10(4t + 1)$ $4t + 1 = 2$ $t = \frac{1}{4}$ h
c **1** Find the second derivative of $x(t)$. $a(t) = v'(t) = x''(t)$	$v(t) = 20(4t + 1)^{-1}$ $v'(t) = -20(4t + 1)^{-2} \times 4$ $a(t) = \frac{-80}{(4t + 1)^2}$
2 Substitute $t = 2$ into $a(t)$.	$a(2) = \frac{-80}{(4(2) + 1)^2}$ $a(2) = \frac{-80}{81}$ km/h^2

EXERCISE 7.2 Applications of derivatives of the natural logarithmic function

ANSWERS p. 401

Recap

1 Find $\frac{dy}{dx}$ for each of the natural logarithmic functions below.

a $y = \ln(5 - 2x)$

b $y = \ln(x^3 + x^2)$

2 Find the second derivative of the function $y = \ln(x + 6)$.

Mastery

3 WORKED EXAMPLE 6 The function $f(x) = \ln(8x - x^2)$ has a stationary point in the interval $0 < x < 8$.

a Find the coordinates of the stationary point.

b Use the second derivative to determine the nature of the stationary point.

4 WORKED EXAMPLE 7 The population of frogs in a wetland is recorded each week for ten weeks. The number of frogs N, after t weeks, is modelled by the function

$$N(t) = 200\ln(-t^2 + 12t + 13).$$

Find

a $N'(t)$

b the number of weeks when the population of frogs is a maximum

c the maximum population of frogs.

5 WORKED EXAMPLE 8 Find the equation of the tangent to the curve $f(x) = \ln(x + e^2)$ at $x = 0$.

6 Find the equation of the tangent to the graph of $y = \ln(x)$ at the point $(3, \ln(3))$.

7 Find the equation of the tangent to the graph of $y = 3\ln(x - 2)$ at the point where the curve crosses the x-axis.

8 WORKED EXAMPLE 9 Given that $\ln(3) \approx 1.0986$, use the increments formula to determine an approximation for $\ln(3.003)$.

9 WORKED EXAMPLE 10 Simon rows a straight stretch of river, for three hours each evening. The distance covered by Simon is given by the function $x(t) = 8\ln(2t + 1)$, where x is the number of kilometres travelled in t hours. Find

a Simon's rowing speed at time t hours

b the number of hours Simon has been rowing when his speed is 4 km/h

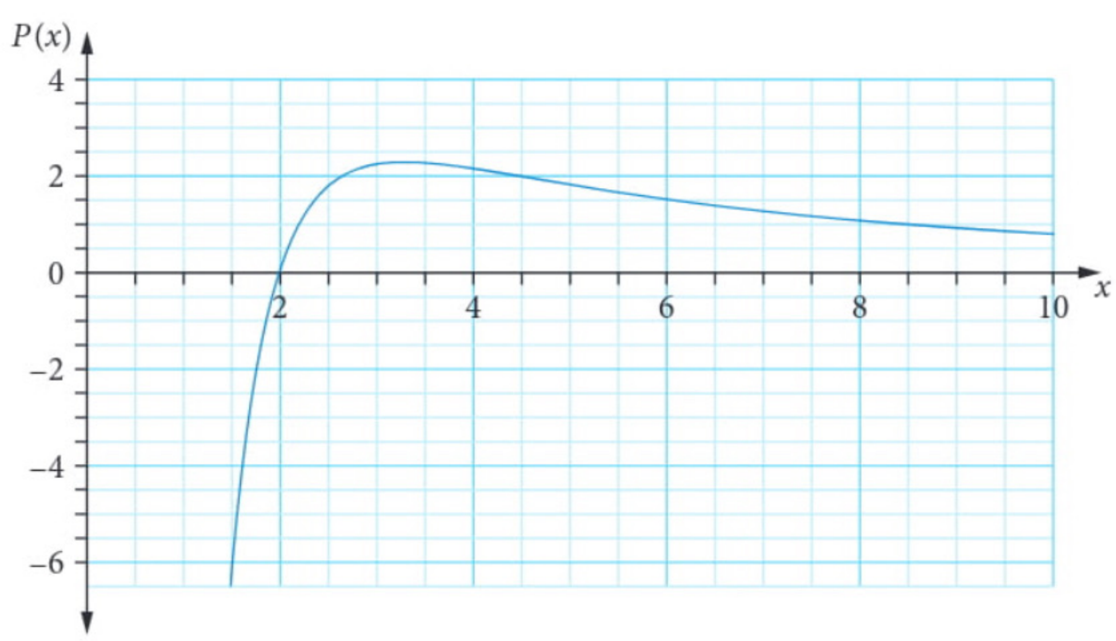
c Simon's acceleration when $t = 1$ hour.

Calculator-free

10 © SCSA MM2018 Q6 (8 marks) A company manufactures and sells an item for $\$x$. The profit, $\$P$, made by the company per item sold is dependent on the selling price and can be modelled by the function

$$P(x) = \frac{50\ln\left(\frac{x}{2}\right)}{x^2} \text{ where } 1.5 \le x \le 10$$

The graph of $P(x)$ is shown below:



a Describe how the profit per item sold varies as the selling price changes. (3 marks)

b Determine the exact price that should be charged for the item if the company wishes to maximise the profit per item sold. (5 marks)

11 © SCSA MM2021 Q1b (3 marks) Let $f'(x) = x \ln(2x)$. Determine a simplified expression for the rate of change of $f'(x)$.

12 © SCSA MM2021 Q3 (3 marks) Given that $\ln(2) \approx 0.693$, use the increments formula to determine an approximation for $\ln(2.02)$.

Calculator-assumed

13 © SCSA MM2016 Q13ab (5 marks)

a Determine $\frac{d}{dx}(x^2 \ln x)$. (2 marks)

b Using your answer from part **a**, show that the graph of $y = x^2 \ln x$ has only one stationary point. (3 marks)

14 (9 marks) The distance covered by a marathon runner in a training run is given by the function $x(t) = \frac{18 \ln(2t+1)}{5}$, where x is the number of kilometres travelled in t hours.

Find

a the speed in terms of t (2 marks)

b the acceleration in terms of t (2 marks)

c the runner's acceleration after 2 hours (2 marks)

d after how many hours will the runner be slowing down at a rate of 1 km/h. Give your answer in hours and minutes, to the nearest minute. (3 marks)

Video playlist Integrals producing natural logarithmic functions

Worksheet Integration of $y = \frac{1}{x}$

7.3 Integrals producing natural logarithmic functions

Integration of reciprocal functions

A **reciprocal function** is a fraction where the variable x, appears only in the denominator.

$$\frac{d}{dx}\ln(x) = \frac{1}{x}$$

So, if we integrate both sides of the equation

$$\int \frac{d}{dx}\ln(x)\,dx = \int \frac{1}{x}\,dx$$

$$\int \frac{1}{x}\,dx = \ln(x) + c$$

Note that this integral is only defined for $x > 0$ because this is the domain of $\ln(x)$.

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For the case where $x < 0$, $-x > 0$ so $\ln(-x)$ is defined.

$$\frac{d}{dx}(\ln(-x)) = \frac{1}{-x} \times -1 \quad \text{by chain rule}$$

$$\frac{d}{dx}(\ln(-x)) = \frac{1}{x}$$

$$\int \frac{1}{x}\,dx = \ln(-x) + c, \text{ where } x < 0.$$

We can summarise this as $\int \frac{1}{x}\,dx = \begin{cases} \ln(x) + c, & x > 0 \\ \ln(-x) + c, & x < 0 \end{cases}$

In the Methods course we only consider the case where the denominator is positive.

The integral of $\frac{1}{x}$

$$\int \frac{1}{x}\,dx = \ln(x) + c, \text{ where } x > 0$$

WORKED EXAMPLE 11 Integrating a simple reciprocal function

Find $\int \frac{4}{7x}\,dx,\ x > 0$.

Steps	Working
1 Factorise by taking out the constant $\frac{4}{7}$.	$\int \frac{4}{7x}\,dx = \frac{4}{7}\int \frac{1}{x}\,dx$
2 Use $\int \frac{1}{x}\,dx = \ln(x) + c$.	$= \frac{4}{7}\ln(x) + c$

Integrating $y = \frac{f'(x)}{f(x)}$

In section 7.1, we used the chain rule to find the derivative below.

$$\frac{d}{dx}\ln(f(x)) = \frac{f'(x)}{f(x)}$$

If we integrate both sides

$$\int \frac{d}{dx}\ln(f(x))\,dx = \int \frac{f'(x)}{f(x)}\,dx$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln(f(x)) + c$$

Integral of $\frac{f'(x)}{f(x)}$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln(f(x)) + c \text{ for } f(x) > 0$$

When integrating a function of the form $\frac{g(x)}{f(x)}$, test whether the numerator, $g(x)$, is a multiple of the derivative of the denominator, $f'(x)$. If this is the case, the integral will be a natural logarithmic function.

WORKED EXAMPLE 12 Integrals of the form $\int \frac{f'(x)}{f(x)}\,dx$ where $f(x) > 0$

Find each integral.

a $\int \frac{2x-3}{x^2-3x+5}\,dx$ where $x^2 - 3x + 5 > 0$

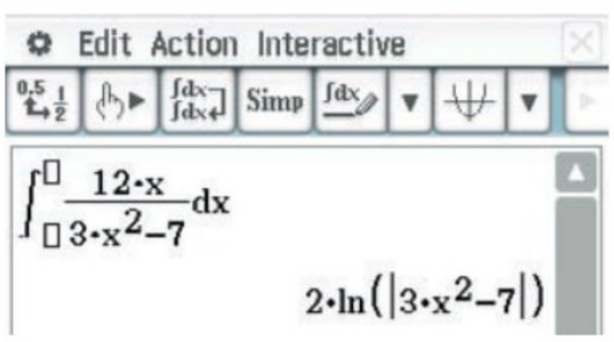
b $\int \frac{12x}{3x^2-7}\,dx$ where $3x^2 - 7 > 0$

Steps	Working
a 1 Find the derivative of the denominator, $f(x)$.	$f(x) = x^2 - 3x + 5$ $f'(x) = 2x - 3$
2 As this derivative is equal to the numerator, write the integral in the form $\int \frac{f'(x)}{f(x)}\,dx = \ln(f(x)) + c$.	$\int \frac{2x-3}{x^2-3x+5}\,dx = \ln(x^2 - 3x + 5) + c$ The restriction $x^2 - 3x + 5 > 0$ ensures the natural logarithmic function is defined.
b 1 Find the derivative of the denominator, $f(x)$.	$f(x) = 3x^2 - 7$ $f'(x) = 6x$
2 As this derivative is equal to a multiple of the numerator, write the integral in the form $\int \frac{f'(x)}{f(x)}\,dx = \ln(f(x)) + c$.	$\int \frac{6x}{3x^2-7}\,dx = \ln(3x^2 - 7)$
3 Multiply both sides by 2 and include the constant in your answer.	$\int \frac{12x}{3x^2-7}\,dx = 2\ln(3x^2 - 7) + c$

USING CAS 2 Finding integrals that produce a natural logarithmic function

Find $\int \frac{12x}{3x^2-7}\,dx$.

ClassPad



1 Enter and highlight the expression.
2 Tap **Interactive** > **Calculus** > ∫.
3 Tap **OK**.

TI-Nspire

1 Press **menu > calculus > integral**.
2 Enter the expression, including the **dx**.

$\int \frac{12x}{3x^2-7}\,dx = 2\ln(3x^2 - 7) + c$

Exam hack

$|3x^2 - 7|$ means the absolute value or modulus of $3x^2 - 7$.

The modulus of a value is its magnitude and can never be negative. This ensures that the natural logarithmic function is always defined, as $\ln(x)$ is not defined when x is negative. It is not necessary to include this modulus sign in your exam answers as this is beyond the scope of the course.

Integration by recognition

Integration by recognition uses the derivative of a function to find the anti-derivative.

7.3

WORKED EXAMPLE 13 Integration by recognition

Find the first derivative of $2x\ln(2x)$ and hence find $\int \ln(2x)\,dx$ where $x > 0$.

Steps	Working
1 Find the first derivative of $2x\ln(2x)$ using the product rule.	$u = 2x$ $\quad v = \ln(2x)$ $\frac{du}{dx} = 2$ $\quad \frac{dv}{dx} = \frac{1}{x}$ $\frac{dy}{dx} = 2x \times \frac{1}{x} + 2\ln(2x)$ $\frac{dy}{dx} = 2 + 2\ln(2x)$
2 Write as a derivative equation and integrate both sides.	$\frac{d}{dx}(2x\ln(2x)) = 2 + 2\ln(2x)$ $\int \frac{d}{dx}(2x\ln(2x))\,dx = \int 2 + 2\ln(2x)\,dx$
3 Simplify the equation.	$2x\ln(2x) = \int 2\,dx + 2\int \ln(2x)\,dx$ $2x\ln(2x) = 2x + 2\int \ln(2x)\,dx$
4 Transpose so that $\int \ln(2x)\,dx$ is the subject of the equation.	$2\int \ln(2x)\,dx = 2x\ln(2x) - 2x$ $\int \ln(2x)\,dx = x\ln(2x) - x + c$

WORKED EXAMPLE 14 Integrating $\frac{1}{ax+b}$ where $x > -\frac{b}{a}$

Find $\int \frac{4}{12x+5}\,dx$ where $x > -\frac{5}{12}$.

Steps	Working
1 Find the derivative of the denominator, $f(x)$.	$f(x) = 12x + 5$ $f'(x) = 12$
2 As this derivative is equal to a multiple of the numerator, write the integral in the form $\int \frac{f'(x)}{f(x)}\,dx = \ln(f(x)) + c$.	$\int \frac{12}{12x+5}\,dx = \ln(12x+5)$
3 Multiply both sides by $\frac{1}{3}$.	$\frac{1}{3}\int \frac{12}{12x+5}\,dx = \frac{1}{3}\ln(12x+5) + c$ $\int \frac{4}{12x+5}\,dx = \frac{1}{3}\ln(12x+5) + c$

WORKED EXAMPLE 15 Evaluating definite integrals

Evaluate $\int_3^5 \frac{1}{x-2}\,dx$.

Steps	Working
1 Find the integral.	$\int_3^5 \frac{1}{x-2}\,dx = [\ln(x-2)]_3^5$
2 Evaluate the integral. Remember, $\ln(1) = 0$.	$= \ln(5-2) - \ln(3-2)$ $= \ln(3) - \ln(1)$ $= \ln(3)$

WORKED EXAMPLE 16 Finding $f(x)$ given $f'(x)$ and a point

Find the equation of the curve $f(x)$ given that $f'(x) = \frac{2}{2x+7}$ where $x > -\frac{7}{2}$ and the curve passes through $(1,0)$.

Steps	Working
1 Integrate $f'(x)$ to find $f(x)$.	$f'(x) = \frac{2}{2x+7}$ $\int f'(x)\,dx = \int \frac{2}{2x+7}\,dx$ $f(x) = \int \frac{2}{2x+7}\,dx$
2 Use the formula $\int \frac{f'(x)}{f(x)}\,dx = \ln(f(x)) + c$.	$f(x) = \ln(2x+7) + c$
3 Substitute the coordinates $(1,0)$ to find the constant, c. Write the function $f(x)$.	$0 = \ln(2 \times 1 + 7) + c$ $c = -\ln(9)$ $f(x) = \ln(2x+7) - \ln(9)$ $f(x) = \ln\left(\frac{2x+7}{9}\right),\ x > -\frac{7}{2}$

WORKED EXAMPLE 17 Finding an unknown pronumeral

Given that $\int_2^k \left(\frac{2}{2x+5}\right)dx = 7$, find the value of k.

Steps	Working
1 Find the integral.	$\int_2^k \left(\frac{2}{2x+5}\right)dx = [\log_e(2x+5)]_2^k$ $= \log_e(2k+5) - \log_e(2(2)+5)$ $= \log_e(2k+5) - \log_e(9)$ $= \log_e\left(\frac{2k-5}{9}\right)$
2 Make this equal to 7 and solve.	$\log_e\left(\frac{2k+5}{9}\right) = 7$ $\frac{2k+5}{9} = e^7$ $2k+5 = 9e^7$ $2k = 9e^7 - 5$ $k = \frac{9e^7-5}{2}$

EXERCISE 7.3 Integrals producing natural logarithmic functions

ANSWERS p. 402

7.3

Recap

1 Find $f'(x)$ given $f(x) = e^x \ln(x)$.

2 For $f(x) = \log_e(x^3 + 1)$, find $f'(2)$.

Mastery

3 WORKED EXAMPLE 11 Find each integral for $x > 0$.

a $\int \frac{2}{x}\,dx$ **b** $\int \frac{6}{5x}\,dx$ **c** $\int \frac{1}{3x}\,dx$

4 WORKED EXAMPLE 12 Find each integral.

a $\int \frac{2x+11}{x^2+11x-15}\,dx$ for $x^2 + 11x - 15 > 0$ **b** $\int \frac{15x^2}{x^3-13}\,dx$ for $x^2 - 13 > 0$

c $\int \frac{18x^2+16x}{3x^3+4x^2+1}\,dx$ for $3x^3 + 4x^2 + 1 > 0$

5 Using CAS 2 Find each integral.

a $\int \frac{1}{x^2-11x+30}\,dx$ for $x^2 - 11x + 30 > 0$ **b** $\int \frac{3}{4x^2-25}\,dx$ for $4x^2 - 25 > 0$

6 WORKED EXAMPLE 13

a Find the first derivative of $f(x) = x^2\log_e(2x)$ where $x > 0$.

b Hence, find $\int x\log_e(2x)\,dx$.

7 a Find the first derivative of $f(x) = x\log_e(x^3)$ where $x > 0$.

b Hence, find $\int \log_e(x^3)\,dx$.

8 Find each integral.

a $\int \frac{1}{5x+3}\,dx$ for $x > -\frac{3}{5}$ **b** $\int \frac{3}{2x-5}\,dx$ for $x > \frac{5}{2}$

9 WORKED EXAMPLE 15 Evaluate each definite integral.

a $\int_1^5 \frac{1}{x}\,dx$ **b** $\int_2^9 \frac{1}{x-1}\,dx$ **c** $\int_6^7 \frac{1}{3x-2}\,dx$ **d** $\int_2^4 \frac{1}{20-3x}\,dx$ **e** $\int_e^{4e} \frac{1}{x}\,dx$

10 WORKED EXAMPLE 16 Find the equation of the curve $f(x)$ given that $f'(x) = \frac{7}{3x-5}$ where $x > \frac{5}{3}$ and $f(2) = 7$.

11 Find the equation of the curve $f(x)$ given that $f'(x) = \frac{9}{x-3} + 4$ where $x > 3$ and $f(4) = 5$.

12 WORKED EXAMPLE 17 Given that $\int_2^m \frac{3}{3x-1}\,dx = 7$, find the value of m.

13 Given that $\int_k^4 \frac{-1}{5-x}\,dx = \ln(2)$, find the value of k.

Calculator-free

14 (4 marks) Let $y = x\log_e(3x)$ where $x > 0$.

a Find $\frac{dy}{dx}$. (2 marks)

b Hence, calculate $\int_1^2 (\log_e(3x) + 1)\,dx$. Express your answer in the form $\log_e(a)$, where a is a positive integer. (2 marks)

15 (5 marks)

a Let $\int_4^5 \frac{2}{2x-1}\,dx = \log_e(b)$. Find the value of b. (2 marks)

b Find p given that $\int_2^3 \frac{1}{1-x}\,dx = \log_e(p)$. (3 marks)

16 © SCSA MM2018 Q7 (6 marks)

a Determine a simplified expression for $\frac{d}{dx}(x\ln(x))$. (2 marks)

b Use your answer from part **a** to show that $\int \ln(x)\,dx = x\ln(x) - x + c$, where c is a constant. (4 marks)

Calculator-assumed

17 (7 marks) The function $f(x)$ has the first derivative $f'(x) = \frac{x+5}{x-1}$, where $x > 1$ and $f(2) = 1$.

a If $f'(x) = a + \frac{b}{x-1}$, show that $a = 1$ and $b = 6$. (1 mark)

b Find $f(x)$. (3 marks)

c Find the gradient of $f(x)$ at $x = 2$. (1 mark)

d Find the equation of the tangent to $f(x)$ at $x = 2$. (2 marks)

7.4 Applications of anti-differentiation involving natural logarithms

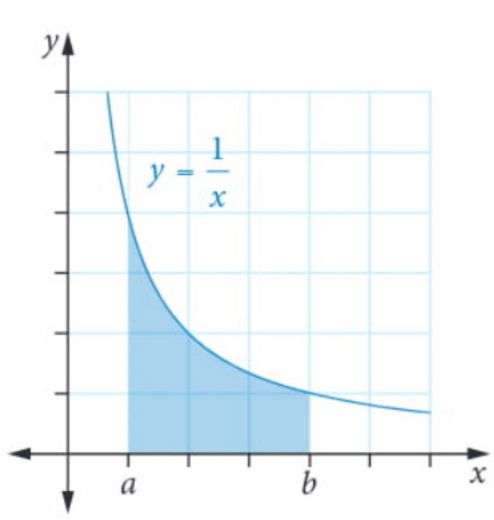
Video playlist Applications of anti-differentiation involving natural logarithms

The area between a curve and the x-axis

The area bounded by the curve $y = \frac{1}{x}$, the x-axis and the lines $x = a$ and $x = b$ is given by the integral equation:

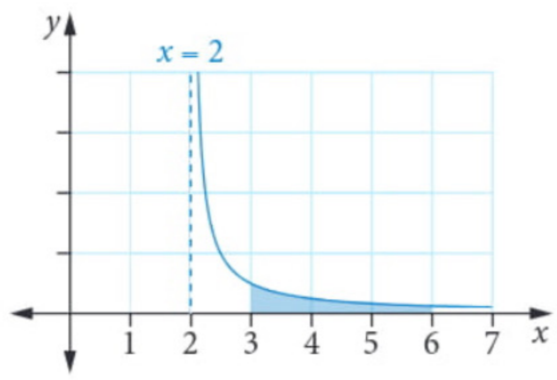
$$\text{area} = \int_a^b \frac{1}{x}\,dx$$

$$\text{area} = \left[\ln(x)\right]_a^b = \ln(b) - \ln(a) \text{ units}^2$$



Exam hack

Always sketch the graph of the function when calculating the area and write square units or units^2 after evaluating the integral.

WORKED EXAMPLE 17	Calculating the area between a reciprocal function and the x-axis
Find the area bounded by the curve $f(x) = \dfrac{1}{2x-4}$, the x-axis and the lines $x = 3$ and $x = 6$.	
Steps	**Working**
1 Sketch the graph of the function and shade the area described. The vertical asymptote occurs where: $2x - 4 = 0$, $x = 2$	
2 Write an integral equation for the area and evaluate. Use the laws of logarithms to simplify the answer.	$\text{area} = \int_3^6 \frac{1}{2x-4}\,dx$ $= \frac{1}{2}[\ln(2x-4)]_3^6$ $= \frac{1}{2}(\ln(2\times 6-4) - \ln(2\times 3-4))$ $= \frac{1}{2}(\ln(8) - \ln(2))$ $= \frac{1}{2}\ln(4) = \ln(2)\ \text{units}^2$

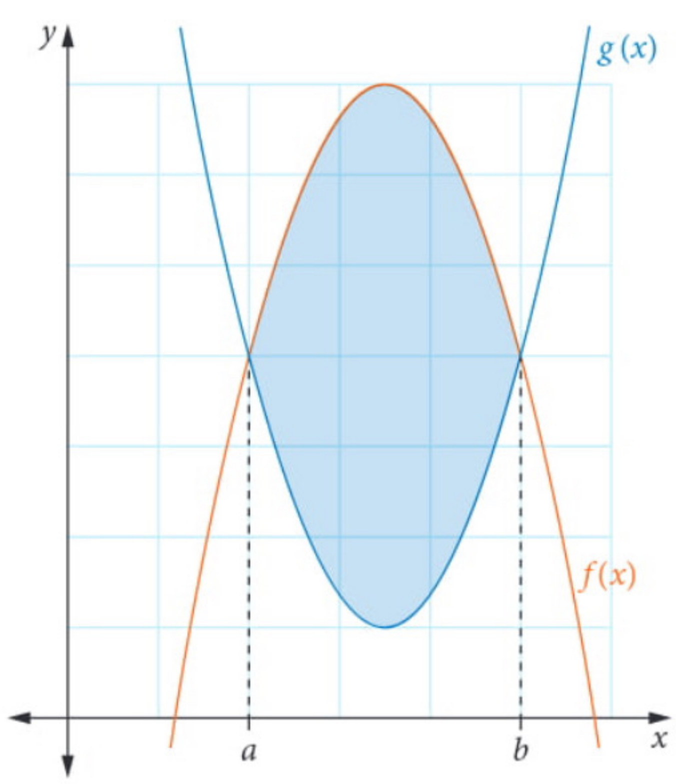
The area bounded by two curves

In the formula given below, $f(x)$ is the upper function and $g(x)$ the lower function. The area is bounded by the functions between the intersection points $x = a$ and $x = b$.

Areas between curves

If $f(x) > g(x)$ for $a < x < b$, then the upper function is $f(x)$ and the lower function is $g(x)$.

$$\text{bounded area} = \int_a^b (\text{upper} - \text{lower})\,dx = \int_a^b [f(x) - g(x)]\,dx$$



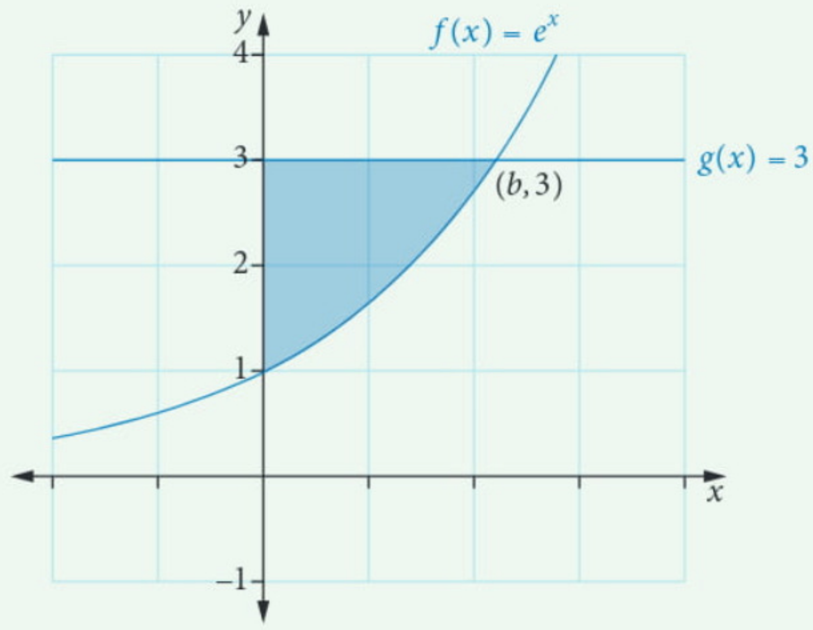
WORKED EXAMPLE 18 Calculating the area bounded by two functions

The functions $f(x) = e^x$ and $g(x) = 3$ intersect at the point $(b, 3)$.

Find

a the exact value of b

b the area bounded by the $f(x)$, $g(x)$ and the y-axis.



Steps	Working
a Solve $f(x) = g(x)$.	$e^x = 3$
Change the equation into exponential form.	$x = \ln(3)$
	$b = \ln(3)$
b 1 Write an integral equation for the area and evaluate.	$\text{area} = \int_0^{\ln 3} (3 - e^x)\, dx$
	$= \left[3x - e^x\right]_0^{\ln 3}$
2 Use the laws of logarithms to simplify the answer.	$= 3\ln(3) - e^{\ln 3} - (0 - e^0)$
3 Use the logarithm law $a^{\log_a b} = b$ to simplify $e^{\ln 3}$.	$= 3\ln(3) - 3 + 1$
	$= 3\ln(3) - 2$ units^2

WACE QUESTION ANALYSIS

Video WACE question analysis: Calculus of the natural logarithmic function

© SCSA MM2020 Q7 Calculator-free (13 marks)

Consider the function $f(x) = e^{2x} - 4e^x$.

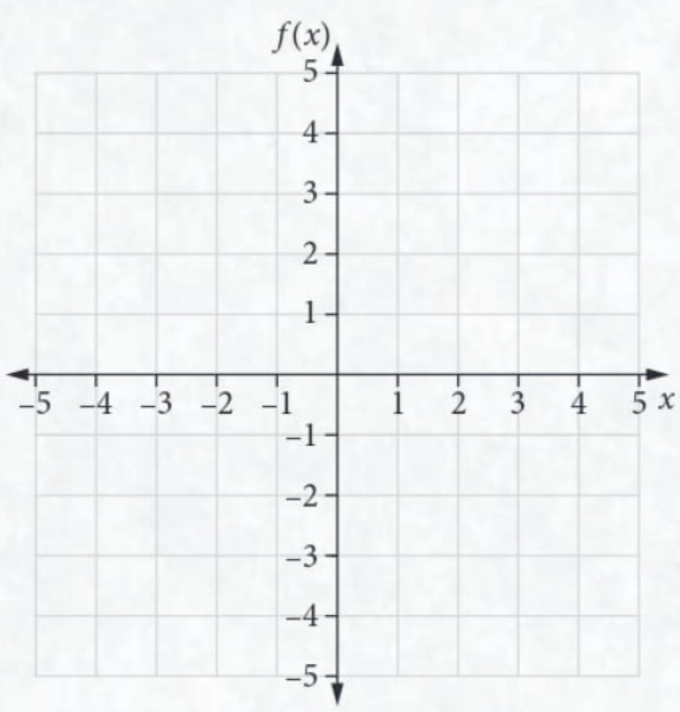
a Determine the coordinates of the x-intercept(s) of f. You may wish to consider the factorised version of f: $f(x) = e^x(e^x - 4)$. (3 marks)

b Show that there is only one turning point on the graph of f, which is located at $(\ln(2), -4)$. (3 marks)

c Determine the coordinates of the point(s) of inflection of f. (3 marks)

d Copy the axes on the right and on them sketch the function f, labelling clearly all intercepts, the turning point and point(s) of inflection. Some approximate values of the natural logarithmic function provided in the table below may be helpful.

x	1	2	3	4
$\ln(x)$	0	0.7	1.1	1.4



(4 marks)

9780170477536

7.4

Reading the question

- Highlight the type of answer required in each part. Where the coordinates are required, you need to find both x and y.
- 'Show' in part **b** indicates that you need to have all working shown. Three marks are allocated, so your working must have at least three parts.
- Highlight the information you should label on your sketched graph. The table of values will assist in getting a better shape.

Thinking about the question

- This function is exponential so solutions to this equation will be natural logarithms.
- You will need to be able to find a first and second derivative.
- You will also need to use the first derivative to find stationary points and the second derivative to find the point(s) of inflection.
- Make sure all the required coordinates are labelled on the graph. You will need to use the table of values to approximate some of your coordinates.

Worked solution (✓ = 1 mark)

a $f(x) = e^x(e^x - 4)$

$\boldsymbol{e^x(e^x - 4) = 0}$ ✓

$e^x - 4 = 0$

$e^x = 4$

$\boldsymbol{x = \ln(4)}$ ✓

x-intercept is $(\ln(4), 0)$. ✓

b $f'(x) = 2e^{2x} - 4e^x$

Solve $f'(x) = 0$.

$0 = 2e^{2x} - 4e^x$

$= 2e^x(e^x - 2)$

$e^x = 2$

$x = \ln(2)$

Substitute $x = \ln(2)$ into $f(x)$:

$f(\ln(2)) = e^{2\ln(2)} - 4e^{\ln(2)}$

$= e^{\ln(4)} - 4e^{\ln(2)}$

$= 4 - 8$

$= -4$

Turning point is at $(\ln(2), -4)$.

differentiates $f(x)$ correctly and equates to 0 ✓

shows the steps required to solve for x ✓

demonstrates the use of log laws to determine the y-coordinate ✓

c $\boldsymbol{f''(x) = 4e^{2x} - 4e^x}$ ✓

$f''(x) = 4e^x(e^x - 1)$

Point of inflection occurs when $f''(x) = 0$.

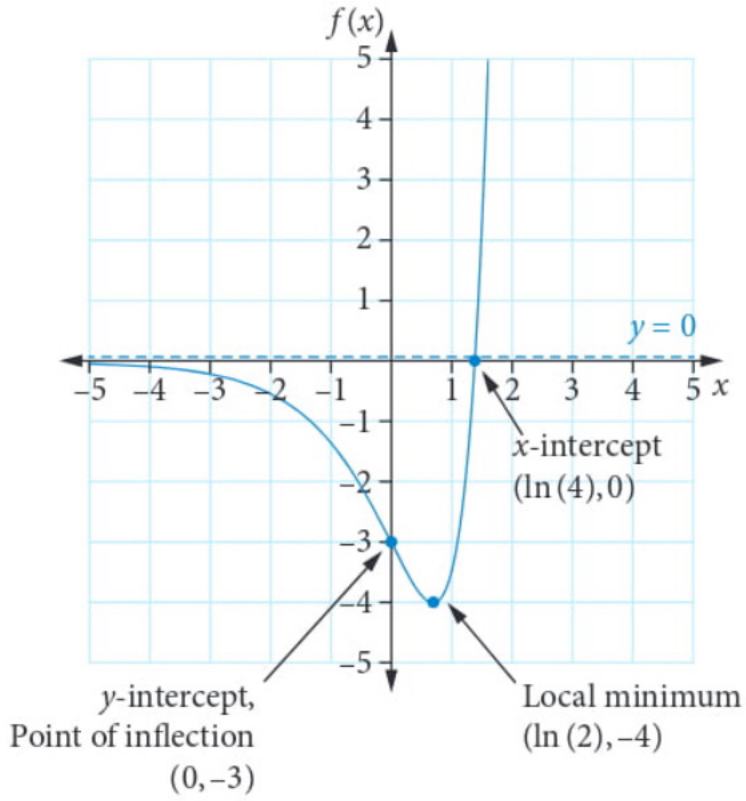
$e^x = 1$

$\boldsymbol{x = \ln(1) = 0}$ ✓

$f(0) = e^0 - 4e^0 = -3$

Point of inflection is at $(0, -3)$. ✓

d



intercepts correct and labelled ✓

turning point and inflection point correct and labelled ✓

concavity correct ✓

limiting behaviour correct ✓

EXERCISE 7.4 Applications involving natural logarithms

ANSWERS p. 402

Recap

1 Given that $\int_0^m \frac{4}{4x+1}\,dx = \ln(13)$ find the value of m.

2 Find the equation of the curve $f(x)$ given that $f'(x) = \frac{1}{x+3}$, $x > -3$ and $f(-2) = 12$.

Mastery

3 WORKED EXAMPLE 17 Find the area bounded by the curve $f(x) = \frac{1}{3x-9}$, the x-axis and the lines $x = 4$ and $x = 5$.

4 Find the area bounded by the curve $f(x) = \frac{4}{x}$, the x-axis and the lines $x = 1$ and $x = e^3$.

5 WORKED EXAMPLE 18 The functions $f(x) = e^x$ and $g(x) = 7$ intersect at the point $(b, 7)$.

Find

a the exact value of b

b the area bounded by the $f(x)$, $g(x)$ and the y-axis.

6 The functions $f(x) = e^x$ and $g(x) = e^2$ intersect at the point (a, e^2).

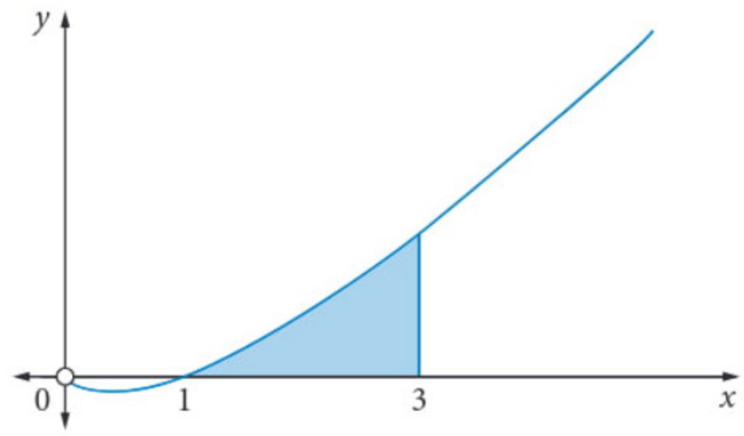
Find

a the value of a

b the area bounded by $f(x)$, the x-axis, the y-axis and the line $x = a$.

Calculator-free

7 (4 marks) Part of the graph of $f: f(x) = x\log_e(x)$ is shown.



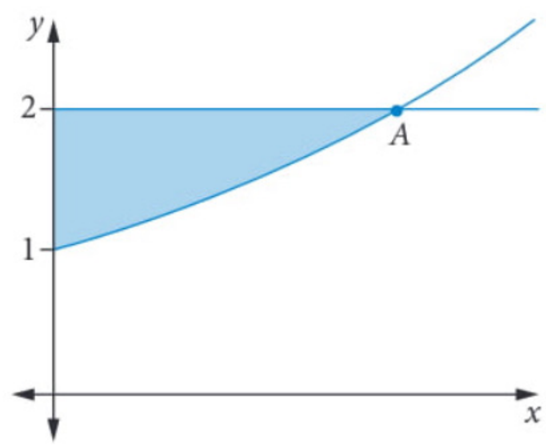
a Find the derivative of $x^2\log_e(x)$. (1 mark)

b Use your answer to part **a** to find the area of the shaded region in the form $a\log_e(b) + c$, where a, b and c are non-zero real constants. (3 marks)

> **Exam hack**
>
> You will need to use integration by recognition to find the required integral.

8 © SCSA MM2017 Q5 (8 marks)

a Consider the shaded area shown between the graph of $y = e^x$, the y axis and the line $y = 2$.



i Determine the coordinates of the point A. (1 mark)

ii Hence or otherwise determine the area between the graph of $y = e^x$, the y axis and the line $y = 2$. (3 marks)

b If the area between the graph of $y = e^x$, the y axis, the x axis and the line $x = k$, where $k \geq 0$, is to be equal to 2 square units, determine the exact value of k. (4 marks)

9 © SCSA MM2019 Q5 (8 marks)

a Determine the area bound by the graph of $f(x) = e^x$ and the x-axis between $x = 0$ and $x = \ln 2$. (3 marks)

b Hence, determine the area bound by the graph of $f(x) = e^x$, the line $y = 2$ and the y-axis. (2 marks)

c Determine the area bound by the graph of $f(x) = e^x$, the line $y = a$ and the y-axis, where a is a positive constant. (3 marks)

Calculator-assumed

10 (4 marks) A small colony of black peppered moths live on a small isolated island. In summer, the population begins to increase. If t is the number of days after 12 midnight on 1 January, the equation that best models the number of moths in the colony at any given time is

$N = 500\ln(21t + 3)$.

a What is the population of black peppered moths on 1 January? (1 mark)

b What is the population of moths after 30 days? (1 mark)

c On which day is the population first greater than 2000? (2 marks)

11 (8 marks) Harrison is training for a 100 m swimming race and wants to get under 50 seconds. At the start, his best time was 1 minute. After 12 days of intensive training, his time has reduced to 55 seconds. Harrison's swim times, T minutes after t days, are modelled using the function $T = 60 - a\ln(t + 1)$.

a Find the value of a, correct to three decimal places. (2 marks)

b How many days will it take him to get under 50 seconds? (2 marks)

c At what rate (in seconds per day, correct to three decimal places) is Harrison's time decreasing at this point? (2 marks)

d How long would it take him to be an Olympic champion contender (under 46 s), assuming his body could stand the training regime? (2 marks)

12 (8 marks) David can currently make about 5 skateboards in a day. He starts to improve his productivity and after two weeks has increased his productivity to 7 skateboards per day. David's daily productivity is modelled using the function $N = k + a\ln(t + 1)$, where t is the number of weeks after starting.

a Find the value of a, correct to three decimal places. (2 marks)

b How long will it take him to get his productivity up to 10 skateboards per day? (2 marks)

c What will be his rate of productivity increase (in skateboards/day) after four weeks? (2 marks)

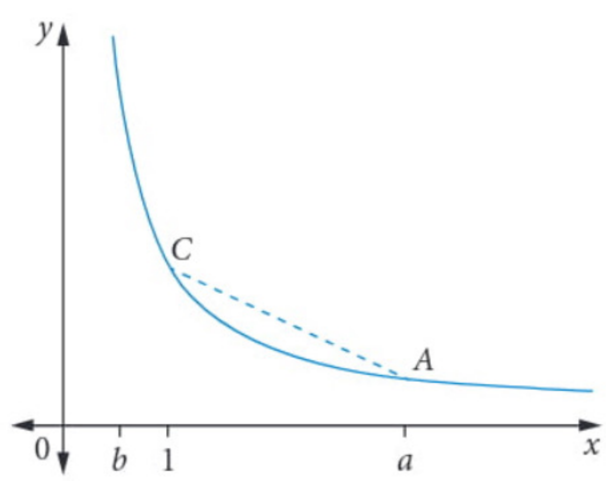
d What will be his rate of productivity increase after ten weeks? (2 marks)

13 © SCSA MM2016 Q13cd (5 marks)

a Sketch the graph of $y = x^2\ln x$, showing all features. (3 marks)

b Calculate the area bounded by the graph of $y = x^2\ln x$, the x axis, $x = 1$ and $x = e$. (2 marks)

14 (12 marks) The diagram shows part of the graph of the function $f(x) = \frac{7}{x}$.



The line segment CA is drawn from the point $C(1, f(1))$ to the point $A(a, f(a))$, where $a > 1$.

a **i** Calculate the gradient of CA in terms of a. (1 mark)

ii At what value of x between 1 and a does the tangent to the graph of f have the same gradient as CA? (2 marks)

b **i** Calculate $\int_1^e f(x)\,dx$. (1 mark)

ii Let b be a positive real number less than one. Find the exact value of b such that $\int_b^1 f(x)\,dx$ is equal to 7. (2 marks)

c **i** Express the area of the region bounded by the line segment CA, the x-axis, the line $x = 1$ and the line $x = a$ in terms of a. (2 marks)

ii For what exact value of a does this area equal 7? (1 mark)

iii Using the value for a determined in **c ii**, explain in words, without evaluating the integral, why $\int_1^a f(x)\,dx < 7$. Use this result to explain why $a < e$. (1 mark)

d Find the exact values of m and n such that $\int_1^{mn} f(x)\,dx = 3$ and $\int_1^{\frac{m}{n}} f(x)\,dx = 2$. (2 marks)

7 Chapter summary

The first derivative of ln(x)

$$\frac{d}{dx}(\ln(x)) = \frac{1}{x}$$

$$\frac{d}{dx}(\ln(ax)) = \frac{1}{x}$$

$$\frac{d}{dx}(\ln(f(x)) = \frac{f'(x)}{f(x)}$$

The laws of logarithms for natural logarithms

$$\ln(xy) = \ln(x) + \ln(y)$$

$$\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$$

$$\ln(x^n) = n\ln(x)$$

Also remember,

$$\ln(1) = 0$$

$$\ln(e^n) = n$$

Stationary points and their nature

- Local maxima occur when $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$.
- Local minima occur when $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$.
- Stationary points of inflection occur when $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 0$ and the concavity of the curve changes from concave up to concave down or from concave down to concave up..

The increments formula

- The increments formula can be used to approximate the increase in the y value for a corresponding small increase in the x value.

  $$\delta y \approx \frac{dy}{dx} \times \delta x$$

  For a given function $y = f(x)$, we can use δy to find an approximation for $f(x + \delta x)$.

  $$f(x + \delta x) \approx f(x) + \delta y$$

Integration of reciprocal functions

- $\int \frac{1}{x}\,dx = \ln(x) + c$ for $x > 0$
- $\int \frac{f'(x)}{f(x)}\,dx = \ln(f(x)) + c$ for $f(x) > 0$

Areas between curves

- If $f(x) > g(x)$ for $a < x < b$, then the upper function is $f(x)$ and the lower function is $g(x)$.

  $$\text{bounded area} = \int_a^b (\text{upper} - \text{lower})\,dx = \int_a^b [f(x) - g(x)]\,dx$$

Cumulative examination: Calculator-free

Total number of marks: 36 Reading time: 4 minutes Working time: 36 minutes

1 (4 marks) The diameter d, in centimetres, of a species of gum tree after t years is given by the rule $d(t) = d_0e^{mt}$. The diameter is 2 cm when the tree is planted, and 10 cm after 2 years.

a Write two equations that can be used to find the constants d_0 and m. (2 marks)

b Calculate the exact values of the constants d_0 and m. (2 marks)

2 (7 marks) A coin is biased so that the probability of tossing a head is p and the probability of tossing a tail is $\frac{2}{3}$. The coin is tossed three times. The discrete random variable X represents the number of tails that occur.

a Find the value of p. (1 mark)

b List the probability distribution of the discrete random variable X. (3 marks)

c Find $P(X \geq 1)$. (1 mark)

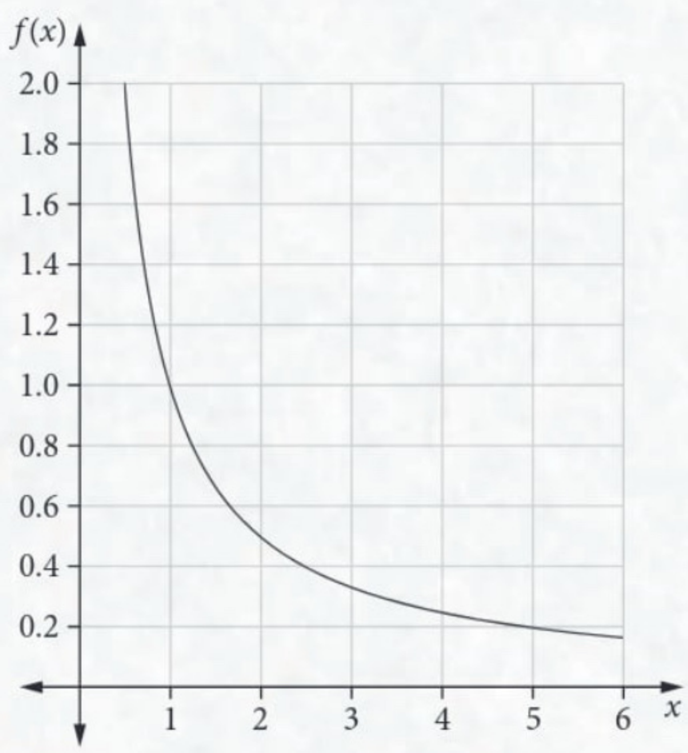
d Find $P(X = 2 \mid X \geq 1)$. (2 marks)

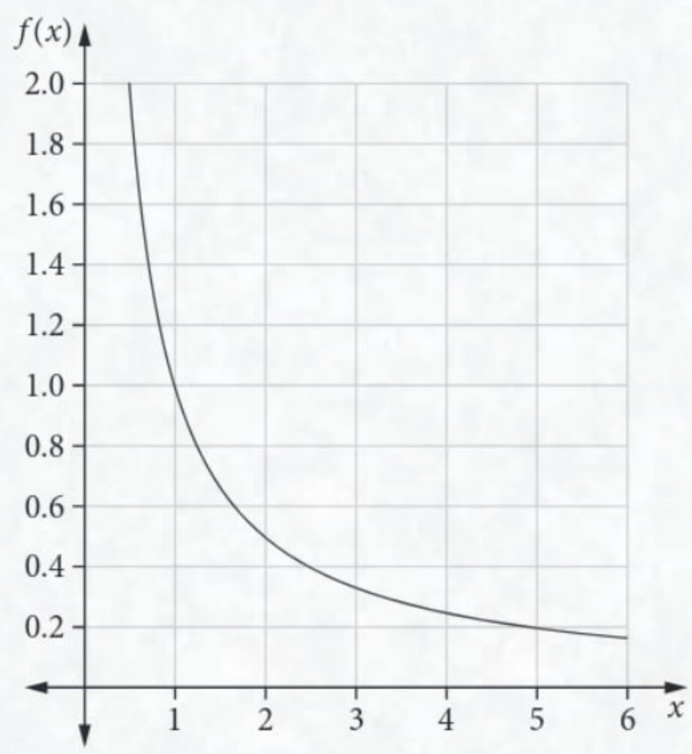
3 (3 marks) Find the coordinates of the x-intercepts of $f(x) = 2e^{2x} - 7e^x + 6$, if the factors of $2e^{2x} - 7e^x + 6$ are $(2e^x - 3)(e^x - 2)$.

4 © SCSA MM2018 Q3cii (3 marks) Evaluate $\int_0^1 \frac{3x+1}{3x^2+2x+1}\,dx$.

5 © SCSA MM2021 Q7 (9 marks)

a Consider the function, $f(x) = \frac{1}{x}$ graphed twice below.

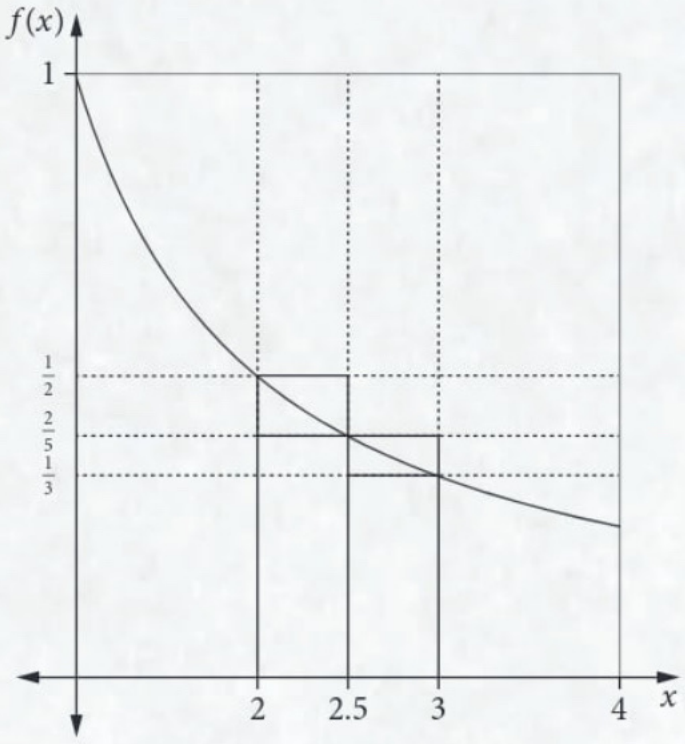




i Copy the graphs and on them shade **two** different regions (one on each graph) each with area exactly ln (2). (2 marks)

ii Given that $\int_a^b \frac{1}{x}\,dx = \ln(3)$, what is the relationship between a and b? (2 marks)

b Another graph of $f(x) = \frac{1}{x}$ is shown below.



i By considering the areas of the rectangles shown, demonstrate and explain why $\frac{11}{30} < \int_2^3 \frac{1}{x}\,dx < \frac{9}{20}$. (3 marks)

ii Hence show that $\frac{11}{30} < \ln(1.5) < \frac{9}{20}$. (2 marks)

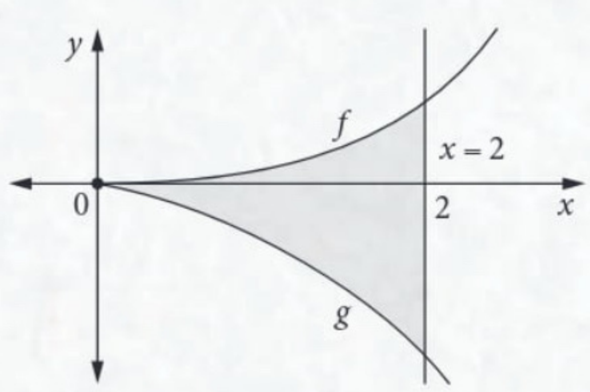
6 (3 marks) The derivative with respect to x of the function $f(x)$ has the rule $f'(x) = \frac{1}{2} - \frac{1}{2x-2}$. Given that $f(2) = 0$, find $f(x)$ in terms of x.

7 (7 marks) Let $f(x) = x^2 e^{kx}$, where k is a positive real constant.

a Show that $f'(x) = xe^{kx}(kx + 2)$. (1 mark)

b Find the value of k for which the graphs of $y = f(x)$ and $y = f'(x)$ have exactly one point of intersection. (2 marks)

Let $g(x) = -\frac{2xe^{kx}}{k}$. The diagram below shows sections of the graphs of f and g for $x \geq 0$.



Let A be the area of the region bounded by the curves $y = f(x)$, $y = g(x)$ and the line $x = 2$.

c Write down a definite integral that gives the value of A. (1 mark)

d Using your result from part **a**, or otherwise, find the value of k such that $A = \frac{16}{k}$. (3 marks)

Cumulative examination: Calculator-assumed

Total number of marks: 29 Reading time: 3 minutes Working time: 29 minutes

1 (10 marks)

Consider the function $f(x) = \frac{1}{27}(ax - 1)^3(b - 3x) + 1$, where a and b are real constants.

a Write down, in terms of a and b, the possible values of x for which $(x, f(x))$ is a stationary point of $f(x)$. (3 marks)

b For what value of a does $f(x)$ have no stationary points? (1 mark)

c Find a in terms of b given that $f(x)$ has one stationary point. (2 marks)

d What is the maximum number of stationary points that $f(x)$ can have? (1 mark)

e Assume that there is a stationary point at $(1, 1)$ and another stationary point (p, p) where $p \neq 1$. Find the value of p. (3 marks)

2 (1 mark) If $\int_1^{12} g(x)\,dx = 5$ and $\int_{12}^{5} g(x)\,dx = -6$, then determine the value of $\int_1^{5} g(x)\,dx$.

3 (9 marks) Consider the function $f(x) = x^4 \ln(4x)$.

a Use the product rule to find $f'(x)$. (2 marks)

b Hence find $\int x^3 \ln(4x)\,dx$. (3 marks)

c Use the result of part **b** to find $\int_{0.25}^{1} x^3 \ln(4x)\,dx$. (2 marks)

An object moves in a straight line with a velocity given by the $v(t) = t^3 \ln(4t)$ m/s.

d Find the distance travelled by the object between $t = 0.25$ s and $t = 1$ s. Give your answer to the nearest centimetre. (2 marks)

4 © SCSA MM2020 Q11 (9 marks)

The line $y = x + c$ is tangent to the graph of $f(x) = e^x$.

a Obtain the coordinates of the point of intersection of the tangent with the graph of $f(x)$. (2 marks)

b What is the value of c? (1 mark)

c Sketch the graph of $f(x)$ and the tangent on the same axes. (1 mark)

d Evaluate the exact area between the graph of $f(x)$, the tangent line, and the line $x = \ln 2$. (3 marks)

e Given that $g(x)$ is the inverse function of $f(x)$, write a definite integral that could be used to determine the area between the graph of $g(x)$, the x-axis, and the line $x = \ln 2$. (2 marks)

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